

# Generalized interpolation in CASL\*

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## Abstract

In this paper we consider the partial many-sorted first-order logic and its extension to the subsorted partial many-sorted first-order logic that underly the CASL specification formalism. First we present counterexamples showing that the generalization of the Craig Interpolation Property does not hold for these logics in general (i.e., with respect to arbitrary signature morphisms). Then we formulate conditions under which the generalization of the Craig Interpolation Property holds for the first logic.

*Keywords:* Formal languages, Formal semantics, Specification languages, Theory of computation, Algebraic specifications, Interpolation property.

## 1 Introduction

There are several structuring mechanisms that have been proposed for algebraic specifications (see [ST 88, Wir 91, SST 92]). Most of them is defined over an arbitrary logical system formalized as an institution (see [GB 92]). For each of the above specification formalisms, there is a sound compositional deduction calculus for reasoning about specifications, which extends the deduction calculus for the underlying logical system. The conditions under which each of these calculi is also complete (relative to the calculus for the underlying logical system) were studied in [Cen 94] and [Borz 97]. The most important and most difficult condition among them is the generalized interpolation property, called  $(\mathcal{D}, \mathcal{T})$ -interpolation in this paper, where  $\mathcal{D}$  presents all morphisms that can be used in the hiding operations, and  $\mathcal{T}$  – morphisms to be used to translate specifications to larger contexts. We are therefore interested in a generalization of the Craig Interpolation Property, where arbitrary  $\mathcal{D}$  and  $\mathcal{T}$  are considered instead of just signature inclusions.

In this paper we consider two logics underlying the CASL (see [CASL 99]) specification formalisms: the partial many-sorted first-order logic and the subsorted partial many-sorted first-order logic. A counterexample we give shows that the generalized interpolation property does not hold in general (i.e., when arbitrary signature morphisms are considered) for both considered logics. We also formulate conditions under which the

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generalized interpolation property holds for the former logic. The situation is worse for the later logic: a counterexample shows that the interpolation does not hold even in its classical version (only signature inclusions are considered).

The proof of a special case of the generalized interpolation property for the partial many-sorted first-order logic is based on the proof of the Craig Interpolation Property for the first-order logic, as presented e.g., in [CK 73] and on the results presented in [Lam 91].

## 2 Definitions

In some definitions presented in this section (especially in Definitions 2.6 and 2.7) we want to abstract away from a particular logical system and consider presented properties for any logical system satisfying certain conditions. A sufficiently abstract (for our purposes) formalization of the notion of a logical system is provided by Goguen and Burstall's notion of an institution (see [GB 92]).

**Definition 2.1 (Institution)** *An institution  $I$  consists of:*

- a category  $\mathbf{Sign}_I$  of signatures;
- a functor  $\mathbf{Sen}_I : \mathbf{Sign}_I \rightarrow \mathbf{Set}$ , giving a set  $\mathbf{Sen}_I(\Sigma)$  of  $\Sigma$ -sentences for each signature  $\Sigma \in |\mathbf{Sign}_I|$ ;
- a functor  $\mathbf{Mod}_I : \mathbf{Sign}_I^{op} \rightarrow \mathbf{DCat}^1$ , giving a discrete category  $\mathbf{Mod}_I(\Sigma)$  of  $\Sigma$ -models for each signature  $\Sigma \in |\mathbf{Sign}_I|$ ;
- for each  $\Sigma \in |\mathbf{Sign}_I|$ , a satisfaction relation  $\models_{\Sigma}^I \subseteq \mathbf{Mod}_I(\Sigma) \times \mathbf{Sen}_I(\Sigma)$  such that for any signature morphism  $\sigma : \Sigma \rightarrow \Sigma'$ ,  $\Sigma$ -sentence  $\varphi \in \mathbf{Sen}_I(\Sigma)$  and  $\Sigma'$ -model  $M' \in \mathbf{Mod}_I(\Sigma')$ :

$$M' \models_{\Sigma'}^I \mathbf{Sen}_I(\sigma)(\varphi) \quad \text{iff} \quad \mathbf{Mod}_I(\sigma)(M') \models_{\Sigma}^I \varphi \quad (\text{Satisfaction condition})$$

□

We will usually write  $\sigma \varphi$  for  $\mathbf{Sen}_I(\sigma)(\varphi)$ ,  $M|_{\sigma}$  for  $\mathbf{Mod}_I(\sigma)(M)$  and  $\models$  for  $\models_{\Sigma}^I$  when it is clear what they mean. In this paper we are especially interested in the following examples of institutions (other examples can be found in [GB 92]):

**Example 2.2** The institution **FOL** of many-sorted first-order logic with equality: *Signatures are many-sorted first-order signatures (with enumerable sets of sort names, operation names and predicate names given with their profiles); sentences are the usual closed formulae of first-order logic built over atomic formulae given either as equalities or atomic predicate formulae; models are the usual many-sorted first-order structures with non-empty carrier sets; satisfaction of a formula in a structure is defined in the standard way.* □

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<sup>1</sup>**DCat** is the category of all discrete categories. For simplicity, we disregard in this paper morphisms between models. Hence, classes of models, rather than model categories are considered.

**Example 2.3** The institution **PFOL** of partial many-sorted first-order logic: *Signatures are partial many-sorted first-order signatures (with enumerable sets of sort names, disjoint sets of total and partial operation names, and predicate names given with their profiles); a signature morphism consists of a mapping between sort names, operation names and predicate names that preserve their profiles; a partial operation symbol may be mapped to a total operation symbol, but not vice versa; sentences are the usual closed formulae of first-order logic built over atomic formulae given either as existential equalities (written  $t_1 \stackrel{e}{=} t_2$ ) or atomic predicate formulae; models are many-sorted first-order structures consisting of many-sorted partial algebras with non-empty carrier sets and with reducts along signature morphisms defined in the usual way; and satisfaction is defined as follows: a predicate symbol applied to a sequence of argument terms holds in a given model iff values of all terms are defined<sup>2</sup> and give a tuple belonging to the interpretation of the predicate symbol, an existential equation  $t_1 \stackrel{e}{=} t_2$  holds iff the values of both terms  $t_1$  and  $t_2$  are defined and identical; this is extended to arbitrary first-order formula in the usual way.*  $\square$

**Example 2.4** The institution **SubPFOL** of subsorted partial many-sorted first-order logic: *Signatures are subsorted partial many-sorted first-order signatures  $\langle S, TF, PF, P, \leq_S \rangle$ , where  $\langle S, TF, PF, P \rangle$  is a **PFOL** signature and  $\leq_S$  is a pre-order relation on the sort names; the pre-order  $\leq_S$  naturally extends to sequences of sorts; we also define overloading relations for operation names as follows:  $f_1 : w_1 \rightarrow s_1 \sim_F f_2 : w_2 \rightarrow s_2$  holds if there exist  $w \in S^*$  and  $s \in S$  such that  $w \leq_S w_1, w_2$ ,  $s_1, s_2 \leq_S s$  and  $f_1 \equiv f_2$ , similarly we define  $\sim_P$  for predicates; subsorted signature morphisms are the usual signature morphisms (as in the institution **PFOL**) that preserve subsort and overloading relations; sentences are usual **PFOL** sentences, except that for every  $s_1, s_2 \in S$  such that  $s_1 \leq_S s_2$  we can use implicit subsort embeddings  $em_{s_1, s_2} : s_1 \rightarrow s_2$ , explicit partial projections  $pr_{s_2, s_1} : s_2 \rightarrow s_1$  and membership predicates  $in_{s_1, s_2}$ ; models are the usual **PFOL** models such that interpretations of subsort embeddings, partial projections and membership predicates satisfy the obvious compatibility conditions (see also [CASL 99]); the satisfaction relation is defined as in the **PFOL** institution.*  $\square$

Institutions **PFOL** and **SubPFOL**, defined above, are institutions underlying the CASL specification framework (see [Mosse 97] and [CASL 99]). The **PFOL** institution is also an example of a many-sorted *negative free logic* (see [Lam 91] for details).

In many specification formalisms signature morphisms are used at least in two ways: the first is to hide some symbols in the (target) specification and, the second, to add and/or rename some symbols in the (source) signature. According to this observation, in each institution  $I$ , we distinguish two classes of signature morphisms: a class  $\mathcal{D}_I$  of signature morphisms appropriate for hiding symbols and, a class  $\mathcal{T}_I$  of signature morphisms appropriate for adding and renaming symbols.

The above observation, plus some technical conditions, are formally expressed by the following definition (see also [Borz 97]).

**Definition 2.5 (( $\mathcal{D}, \mathcal{T}$ )-institution)** *Let  $\mathcal{D}_I, \mathcal{T}_I \subseteq \mathbf{Sign}_I$  be classes of signature morphisms in an institution  $I$ . We say that the institution  $I$  with distinguished classes of morphisms  $\mathcal{D}_I$  and  $\mathcal{T}_I$  is ( $\mathcal{D}, \mathcal{T}$ )-institution iff:*

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<sup>2</sup>The value of a term is defined iff values of all its subterms are defined and belong to a proper domain.

- classes  $\mathcal{D}_I$  and  $\mathcal{T}_I$  are closed under composition and include all identities;
- for every  $(d : \Sigma \rightarrow \Sigma_1) \in \mathcal{D}_I$  and  $(t : \Sigma \rightarrow \Sigma_2) \in \mathcal{T}_I$  there exist  $(t' : \Sigma_1 \rightarrow \Sigma') \in \mathcal{T}_I$  and  $(d' : \Sigma_2 \rightarrow \Sigma') \in \mathcal{D}_I$  such that the following diagram is a pushout in  $\mathbf{Sign}_I$ :

$$\begin{array}{ccc}
 & \Sigma & \\
 d \in \mathcal{D}_I \swarrow & & \searrow t \in \mathcal{T}_I \\
 \Sigma_1 & & \Sigma_2 \\
 t' \in \mathcal{T}_I \searrow & & \swarrow d' \in \mathcal{D}_I \\
 & \Sigma' &
 \end{array}$$

□

In the rest of this section we define properties of logical systems, which are crucial for the rest of the paper. The first,  $(\mathcal{D}, \mathcal{T})$ -interpolation property of a logical system, is particularly important for completeness of logical systems for specification formalisms built over them (see [Borz 97]). The  $(\mathcal{D}, \mathcal{T})$ -interpolation property, presented below, is inspired by the institutional generalization of the Craig Interpolation Property presented in [Tar 86].

**Definition 2.6** ( $(\mathcal{D}, \mathcal{T})$ -interpolation) *A  $(\mathcal{D}, \mathcal{T})$ -institution  $I$  satisfies the  $(\mathcal{D}, \mathcal{T})$ -interpolation property iff for all  $d, d' \in \mathcal{D}_I$  and  $t, t' \in \mathcal{T}_I$  that form a pushout in  $\mathbf{Sign}_I$  (as in Definition 2.5), and  $\varphi_1 \in \mathbf{Sen}_I(\Sigma_1)$  and  $\varphi_2 \in \mathbf{Sen}_I(\Sigma_2)$ , if*

$$\mathbf{Sen}_I(t')(\varphi_1) \models_{\Sigma'}^I \mathbf{Sen}_I(d')(\varphi_2)$$

*then there exists  $\varphi \in \mathbf{Sen}_I(\Sigma)$ , called  $(d, t)$ -interpolant of  $\varphi_1$  and  $\varphi_2$ , such that:*

$$\varphi_1 \models_{\Sigma_1}^I \mathbf{Sen}_I(d)(\varphi) \quad \text{and} \quad \mathbf{Sen}_I(t)(\varphi) \models_{\Sigma_2}^I \varphi_2.$$

□

The next property is an extension of the well known amalgamation property to a  $(\mathcal{D}, \mathcal{T})$ -institution.

**Definition 2.7** (weak- $(\mathcal{D}, \mathcal{T})$ -amalgamation) *A  $(\mathcal{D}, \mathcal{T})$ -institution  $I$  satisfies the weak- $(\mathcal{D}, \mathcal{T})$ -amalgamation property iff for all  $d, d' \in \mathcal{D}_I$  and  $t, t' \in \mathcal{T}_I$  that form a pushout in  $\mathbf{Sign}_I$  (as in Definition 2.5) and for any  $M_1 \in \mathbf{Mod}_I(\Sigma_1)$  and  $M_2 \in \mathbf{Mod}_I(\Sigma_2)$ , if  $M_1|_d = M_2|_t$  then there exists a model  $M' \in \mathbf{Mod}_I(\Sigma')$  such that  $M'|_{t'} = M_1$  and  $M'|_{d'} = M_2$ .*

□

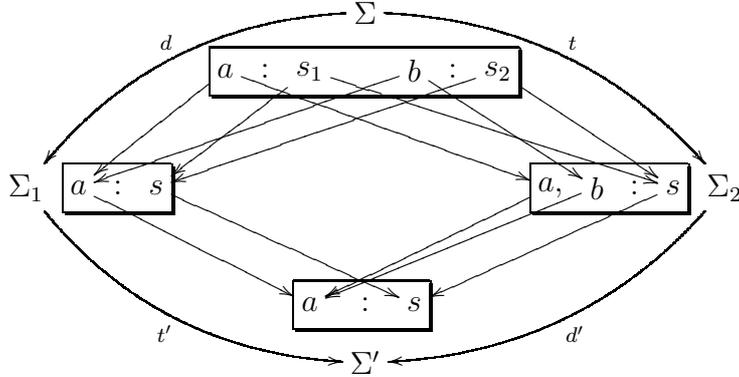
**Fact 2.8** *For any classes  $\mathcal{D}_{\mathbf{PFOL}}, \mathcal{T}_{\mathbf{PFOL}} \subseteq \mathbf{Sign}_{\mathbf{PFOL}}$ , that together with the institution  $\mathbf{PFOL}$  satisfy conditions of Definition 2.5, the  $(\mathcal{D}, \mathcal{T})$ -institution  $\mathbf{PFOL}$  has the weak- $(\mathcal{D}, \mathcal{T})$ -amalgamation property.*

□

### 3 Negative examples

The classical interpolation property for many-sorted first-order logic (see [KK 67]) and partial first-order logic (see [Gum 79, Lam 91]) is a special case of the  $(\mathcal{D}, \mathcal{T})$ -interpolation property, where classes  $\mathcal{D}_I$  and  $\mathcal{T}_I$  are classes of signature injections. The following example shows that the  $(\mathcal{D}, \mathcal{T})$ -interpolation property does not hold for some  $(\mathcal{D}, \mathcal{T})$ -institution **PFOL**, where classes  $\mathcal{D}_{\mathbf{PFOL}}$  and  $\mathcal{T}_{\mathbf{PFOL}}$  contain some non-injective signature morphisms.

**Example 3.1** *Let:  $\Sigma = \langle \{s_1, s_2\}, \{a : s_1, b : s_2\}, \emptyset, \emptyset \rangle$ ,  $\Sigma_1 = \langle \{s\}, \{a : s\}, \emptyset, \emptyset \rangle$ ,  $\Sigma_2 = \langle \{s\}, \{a, b : s\}, \emptyset, \emptyset \rangle$  and  $\Sigma' = \Sigma_1$  be **PFOL** signatures with empty sets of partial operation and predicate names, and  $(d : \Sigma \rightarrow \Sigma_1), (d' : \Sigma_2 \rightarrow \Sigma') \in \mathcal{D}_{\mathbf{PFOL}}$  and  $(t : \Sigma \rightarrow \Sigma_2), (t' : \Sigma_1 \rightarrow \Sigma') \in \mathcal{T}_{\mathbf{PFOL}}$  be signature morphisms such that:  $d(s_1) = d(s_2) = s$ ,  $d(a) = d(b) = a$ ,  $t(s_1) = t(s_2) = s$ ,  $t(a) = a$ ,  $t(b) = b$ ,  $t'(s) = s$ ,  $t'(a) = a$ ,  $d'(s) = s$  and  $d'(a) = d'(b) = a$  and the following diagram is a pushout in  $\mathbf{Sign}_{\mathbf{PFOL}}$ :*



Let us consider the following judgment:  $t'(a \stackrel{e}{=} a) \models_{\Sigma'} d'(a \stackrel{e}{=} b)$  which is satisfied by any  $\Sigma'$ -model  $M'$ . Now, we assume that there exists a  $\Sigma$ -sentence  $\varphi$  such that

$$a \stackrel{e}{=} a \models_{\Sigma_1} d(\varphi) \quad \text{and} \quad t(\varphi) \models_{\Sigma_2} a \stackrel{e}{=} b.$$

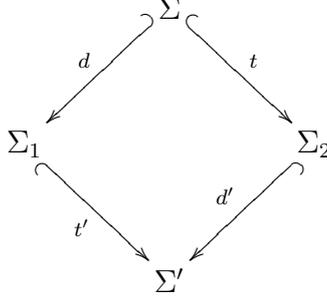
Let  $M_2$  be a  $\Sigma_2$ -model such that the sentence  $a \stackrel{e}{=} b$  does not hold in  $M_2$ . It means that  $|a : s|_{M_2} \neq |b : s|_{M_2}$ . Now, in  $M_2|_t$  we have  $|M_2|_t|_{s_1} = |M_2|_s = |M_2|_t|_{s_2}$  and also  $|a : s_1|_{M_2|_t} \neq |b : s_2|_{M_2|_t}$ . Next, let  $M$  be a  $\Sigma$ -model that is the same as  $M_2|_t$  except that  $|a : s_1|_M = |b : s_2|_{M_2|_t}$ . Finally, let  $M_1$  be the only  $\Sigma_1$ -model such that  $M_1|_d = M$ .

Now, since  $M_1 \models_{\Sigma_1} a \stackrel{e}{=} a$ , we have  $M_1 \models_{\Sigma_1} d(\varphi)$ . By the satisfaction condition we obtain  $M \models_{\Sigma} \varphi$ . Next, because  $M$  and  $M_2|_t$  are isomorphic, we have  $M_2|_t \models_{\Sigma} \varphi$  and by the satisfaction condition  $M_2 \models_{\Sigma_2} t(\varphi)$ , which contradicts  $t(\varphi) \models_{\Sigma_2} a \stackrel{e}{=} b$ , since the sentence  $a \stackrel{e}{=} b$  does not hold in  $M_2$ .  $\square$

It follows from the construction of the above counterexample that the same argument can be repeated for a  $(\mathcal{D}, \mathcal{T})$ -institution **FOL**, since the institution **FOL** can be considered as a “fragment” of the **PFOL** institution and partiality is not used in the example.

The following example shows that the  $(\mathcal{D}, \mathcal{T})$ -interpolation property does not hold for the  $(\mathcal{D}, \mathcal{T})$ -institution **SubPFOL** even if the classes  $\mathcal{D}_{\mathbf{SubPFOL}}$  and  $\mathcal{T}_{\mathbf{SubPFOL}}$  are restricted to all morphisms injective on both sort and operation names.

**Example 3.2** Let  $\Sigma = \langle \{s_1, s_2\}, \{a, b : s_1, c : s_2\}, \emptyset, \emptyset, \emptyset \rangle$  be the **SubPFOL** signature,  $\Sigma_1, \Sigma_2$  and  $\Sigma'$  be **SubPFOL** signatures defined as  $\Sigma$ , except the subsort relation given by  $s_1 \leq_S s_2$ , and  $(d : \Sigma \hookrightarrow \Sigma_1), (d' : \Sigma_2 \hookrightarrow \Sigma') \in \mathcal{D}_{\mathbf{SubPFOL}}$  and  $(t : \Sigma \hookrightarrow \Sigma_2), (t' : \Sigma_1 \hookrightarrow \Sigma') \in \mathcal{T}_{\mathbf{SubPFOL}}$  be signature inclusions such that the following diagram is a pushout in  $\mathbf{Sign}_{\mathbf{SubPFOL}}$ :



Let us consider the following judgment  $t'(em_{s_1, s_2}(a) \stackrel{e}{=} c) \models_{\Sigma'} d'(em_{s_1, s_2}(b) \stackrel{e}{=} c \Rightarrow a \stackrel{e}{=} b)$  which is satisfied by any  $\Sigma'$ -model. We assume that there exists a  $\Sigma$ -sentence  $\varphi$  such that

$$em_{s_1, s_2}(a) \stackrel{e}{=} c \models_{\Sigma_1} d(\varphi) \quad \text{and} \quad t(\varphi) \models_{\Sigma_2} em_{s_1, s_2}(b) \stackrel{e}{=} c \Rightarrow a \stackrel{e}{=} b.$$

Let  $M_1$  be a  $\Sigma_1$ -model given as follows:  $|M_1|_{s_1} = |M_1|_{s_2} = \{v_1, v_2\}$ , where  $v_1$  and  $v_2$  are different values,  $|em_{s_1, s_2}|_{M_1}$  is identity,  $|a : s_1|_{M_1} = |c : s_2|_{M_1} = v_1$  and  $|b : s_1|_{M_1} = v_2$ , and let  $M_2$  be a  $\Sigma_2$ -model with the same carrier sets and interpretations of constants  $a, b : s_1$  and  $c : s_2$  as in  $M_1$  and such that  $|em_{s_1, s_2}|_{M_2}(v_1) = v_2$  and  $|em_{s_1, s_2}|_{M_2}(v_2) = v_1$ .

Now, since  $M_1 \models_{\Sigma_1} em_{s_1, s_2}(a) \stackrel{e}{=} c$  we have  $M_1 \models_{\Sigma_1} d(\varphi)$  which by the satisfaction condition is equivalent to  $M_1|_d \models_{\Sigma} \varphi$ . Next, because  $M_1|_d = M_2|_t$  we obtain  $M_2 \models_{\Sigma_2} t(\varphi)$ . Finally, by assumption  $M_2 \models_{\Sigma_2} em_{s_1, s_2}(b) \stackrel{e}{=} c \Rightarrow a \stackrel{e}{=} b$ , which is not true and contradicts our assumption.  $\square$

Let us notice that the weak- $(\mathcal{D}, \mathcal{T})$ -amalgamation property does not hold for the  $(\mathcal{D}, \mathcal{T})$ -institution **SubPFOL** considered in the above example (there is no amalgamated model for models  $M_1$  and  $M_2$  from the above example). More about the relation between interpolation and (weak) amalgamation can be found in [Gab 71, Maks 77, Makk 95, DM 00].

## 4 Positive results

In this section we formulate conditions, under which the  $(\mathcal{D}, \mathcal{T})$ -interpolation property holds for the  $(\mathcal{D}, \mathcal{T})$ -institution **PFOL**, where classes  $\mathcal{D}_{\mathbf{PFOL}}$  and  $\mathcal{T}_{\mathbf{PFOL}}$  satisfy certain conditions. First, we define the notion of an *extension* of a given **PFOL** signature by constants (see [KK 67, CK 73, Lam 91] for similar notions).

**Definition 4.1** For every signature  $\Sigma \in |\mathbf{Sign}_{\mathbf{PFOL}}|$ , the extension of  $\Sigma$  by constants is the signature  $\Sigma^C \in |\mathbf{Sign}_{\mathbf{PFOL}}|$  with the same sets of sorts, partial operations and predicate names and with the set of total operations extended by a countable set of total constants of sort  $s$ , for every sort  $s$  in  $\Sigma$ .

For every signature morphism  $(\sigma : \Sigma_1 \rightarrow \Sigma_2) \in \mathbf{Sign}_{\mathbf{PFOL}}$  injective on sort names we

define  $(\sigma^C : \Sigma_1^C \rightarrow \Sigma_2^C) \in \mathbf{Sign}_{\mathbf{PFOL}}$  as the following extension of  $\sigma$ : let  $s_1$  be a sort in  $\Sigma_1$  and  $s_2$  be a sort in  $\Sigma_2$  such that  $\sigma(s_1) = s_2$ ,  $C_1 = \{c_1^1 : s_1, c_2^1 : s_1, \dots\}$  and  $C_2 = \{c_1^2 : s_2, c_2^2 : s_2, \dots\}$  be sets of total constants of sort  $s_1$  and  $s_2$  added to signature  $\Sigma_1$  and  $\Sigma_2$ , respectively, then we define<sup>3</sup>  $\sigma^C(c_i^1 : s_1) = c_i^2 : s_2$ , for  $i = 1, 2, \dots$

We also say that a class  $\Theta$  of signature morphisms is closed under extension by constants iff for every signature morphism  $\sigma \in \Theta$ , if  $\sigma$  is injective on sort names then  $\sigma^C \in \Theta$ .  $\square$

**Theorem 4.2 (( $\mathcal{D}, \mathcal{T}$ )-interpolation in  $\mathbf{PFOL}$ )** *Let  $\mathcal{D}_{\mathbf{PFOL}}$  and  $\mathcal{T}_{\mathbf{PFOL}}$  be classes of signature morphisms injective on sort names and closed under extension by constants. Then the  $(\mathcal{D}, \mathcal{T})$ -institution  $\mathbf{PFOL}$  has the  $(\mathcal{D}, \mathcal{T})$ -interpolation property.  $\square$*

The proof of the above theorem is an extension of proofs of a similar result for the single-sorted first-order logic presented in [CK 73] and the model existence theorem for negative variant (i.e. predicates do not hold on non-defined terms) of partial single-sorted first-order logic presented in [Lam 91] to the  $(\mathcal{D}, \mathcal{T})$ -institution  $\mathbf{PFOL}$ <sup>4</sup> (technical details of the proof can be found in [Borz 00]).

The result presented in the above theorem for a  $(\mathcal{D}, \mathcal{T})$ -institution  $\mathbf{PFOL}$ , where classes  $\mathcal{D}_{\mathbf{PFOL}}$  and  $\mathcal{T}_{\mathbf{PFOL}}$  satisfy certain conditions, is also valid for any  $(\mathcal{D}, \mathcal{T})$ -institution  $\mathbf{FOL}$  satisfying similar conditions.

In Example 3.1 we showed that the  $(\mathcal{D}, \mathcal{T})$ -institution  $\mathbf{PFOL}$  (and also  $\mathbf{FOL}$ ) does not have the  $(\mathcal{D}, \mathcal{T})$ -interpolation property, when classes  $\mathcal{D}_{\mathbf{PFOL}}$  and  $\mathcal{T}_{\mathbf{PFOL}}$  contain certain signature morphisms that are non-injective on sort names. From Theorem 4.2 we know that the  $(\mathcal{D}, \mathcal{T})$ -institution  $\mathbf{PFOL}$  has the  $(\mathcal{D}, \mathcal{T})$ -interpolation property when all signature morphisms in both classes  $\mathcal{D}_{\mathbf{PFOL}}$  and  $\mathcal{T}_{\mathbf{PFOL}}$  are injective on sort names. Unfortunately, we do not know whether the  $(\mathcal{D}, \mathcal{T})$ -interpolation holds for  $(\mathcal{D}, \mathcal{T})$ -institutions  $\mathbf{PFOL}$  when only one class of signature morphisms,  $\mathcal{D}_{\mathbf{PFOL}}$  or  $\mathcal{T}_{\mathbf{PFOL}}$  is required to be injective on sort names.

## 5 Conclusions

In this paper we consider the  $(\mathcal{D}, \mathcal{T})$ -interpolation property for two logics underlying the CASL specification formalism: the partial many-sorted first-order logic (formalized as  $(\mathcal{D}, \mathcal{T})$ -institution  $\mathbf{PFOL}$ ) and the subsorted partial many-sorted first-order logic (formalized as  $(\mathcal{D}, \mathcal{T})$ -institution  $\mathbf{SubPFOL}$ ). We proved the  $(\mathcal{D}, \mathcal{T})$ -interpolation property for the  $(\mathcal{D}, \mathcal{T})$ -institution  $\mathbf{PFOL}$ , where both classes of morphisms,  $\mathcal{D}$  and  $\mathcal{T}$ , are injective on sort names. A counterexample shows that the  $(\mathcal{D}, \mathcal{T})$ -interpolation property does not hold for the  $(\mathcal{D}, \mathcal{T})$ -institution  $\mathbf{PFOL}$ , when both classes,  $\mathcal{D}$  and  $\mathcal{T}$ , contain some non-injective signature morphisms. An interesting, and still open, problem is the

<sup>3</sup>For a given signature morphism  $\sigma$ ,  $\sigma^C$  is not unique. Uniqueness can be obtained e.g., by introducing an order of type  $\omega$  on sort names and on (added) total constants.

<sup>4</sup>The proof is similar to the proof of the classical interpolation property presented in [CK 73]. The main points are: non-emptiness of carrier sets, inseparability of sets of sentences considered along  $(\mathcal{D}, \mathcal{T})$ -pushouts and extensions of signatures by constants (see Definition 4.1) considered instead of “witnesses construction” considered in [CK 73].

$(\mathcal{D}, \mathcal{T})$ -interpolation property for the  $(\mathcal{D}, \mathcal{T})$ -institution **PFOL**, when only one class of signature morphisms is a class of morphisms injective on sort names.

Another interesting, and still open, problem is the  $(\mathcal{D}, \mathcal{T})$ -interpolation property for the  $(\mathcal{D}, \mathcal{T})$ -institution **SubPFOL**. By a counterexample we show that the  $(\mathcal{D}, \mathcal{T})$ -interpolation property does not hold for the  $(\mathcal{D}, \mathcal{T})$ -institution **SubPFOL** even if both classes of signature morphisms are injective. It means that **SubPFOL** does not have the *classical* interpolation property.

A task for the future is to propose conditions under which the  $(\mathcal{D}, \mathcal{T})$ -institution **SubPFOL** and its extension by sort generation sentences (see [CASL 99]) have the  $(\mathcal{D}, \mathcal{T})$ -interpolation property. Another interesting task is to study the  $(\mathcal{D}, \mathcal{T})$ -interpolation property for a  $(\mathcal{D}, \mathcal{T})$ -institution **SubPFOL** with injective classes  $\mathcal{D}_{\text{SubPFOL}}$  and  $\mathcal{T}_{\text{SubPFOL}}$ , and such that the  $(\mathcal{D}, \mathcal{T})$ -institution **SubPFOL** has the weak- $(\mathcal{D}, \mathcal{T})$ -amalgamation property.

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